LESSON 14

TRIGONOMETRIC IDENTITIES

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, shape and measurement
Assessment Standard

Derive and use the following identities:

- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\sin^2\theta + \cos^2\theta = 1$

Overview

In this lesson you will:

- Use *x*, *y* and *r* to derive the above two identities.
- Use the above identities to simplify trigonometric expressions.
- Use the above identities to prove more complicated trigonometric identities.

Lesson

To simplify or prove trig expressions or identities, we need to change everything to sin θ and/or cos θ.

1. $\tan\theta = \frac{y}{x}$

$$\text{and } \frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}}$$

$$= \frac{y}{r} \times \frac{r}{x}$$

$$= \frac{y}{x}$$

$$\text{so } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

2. The magic **ONE** of trigonometry

By Pythagoras $x^2 + y^2 = r^2$

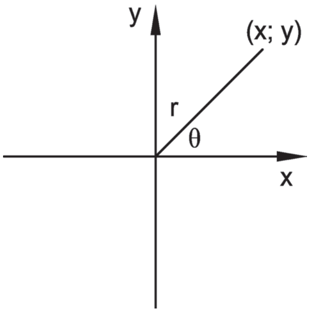
$\div r^2$ $\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$

but $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$

so $\cos^2\theta + \sin^2\theta = 1$

Very important: $\sin^2\theta = 1 - \cos^2\theta$

$\cos^2\theta = 1 - \sin^2\theta$



Let's use what we have learnt

1. Simplify: $\frac{1}{\cos^2\theta}(1 - \sin^2\theta)$

Solution

$\frac{1}{\cos^2\theta} \times \cos^2\theta = 1$ (Using $1 - \sin^2\theta = \cos^2\theta$)

2. Prove: $\frac{\sin\theta}{1+\cos\theta}+\frac{1}{\tan\theta}=\frac{1}{\tan\theta\times\cos\theta}$

Solution

When we prove an identity we have to show that what is on the LHS is exactly what is written on the RHS. So sometimes we need to simplify both to show this. Remember to keep the sides separate.

Prove that $\frac{1}{\cos x}+\tan x=\frac{\cos x}{1-\sin x}$

LHS: $\frac{1}{\cos x}+\frac{\sin x}{\cos x}$

$=\frac{1+\sin x}{\cos x}\times\frac{1-\sin x}{1-\sin x}$

$=\frac{1-\sin^2 x}{\cos x(1-\sin x)}$

$=\frac{\cos^2 x}{\cos x(1-\sin x)}=\frac{\cos x}{1-\sin x}$

$\therefore$ LHS $\equiv$ RHS

3. Prove that $\frac{\sin\theta\cdot\tan\theta}{1-\cos\theta}-1=\frac{1}{\cos\theta}$

Proof:

LHS $=\frac{\sin\theta\cdot\frac{\sin\theta}{\cos\theta}}{1-\cos\theta}-1$

$=\frac{\sin^2\theta}{\cos\theta(1-\cos\theta)}-1$

$=\frac{(1-\cos^2\theta)}{\cos\theta(1-\cos\theta)}-1$

$=\frac{(1-\cos\theta)(1+\cos\theta)}{\cos\theta(1-\cos\theta}-1$

$=\frac{1+\cos\theta}{\cos\theta}-\frac{\cos\theta}{\cos\theta}$

$=\frac{1+\cos\theta-\cos\theta}{\cos\theta}$

$=\frac{1}{\cos\theta}$

= RHS

4. Prove $\left(\frac{1}{\cos x}-\tan x\right)^2=\frac{1-\sin x}{1+\sin x}$

Solution

LHS $\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right)^2$ ($\tan x=\frac{\sin x}{\cos x}$)

$=\left(\frac{1-\sin x}{\cos x}\right)^2$ (LCD $\cos x$)

$=\frac{(1-\sin x)^2}{1-\sin^2 x}$ ($\cos^2 x=1-\sin^2 x$)

$=\frac{(1-\sin x)^2}{(1-\sin x)(1+\sin x)}$ ($a^2-b^2=(a-b)(a+b)$)

$=\frac{1-\sin x}{1+\sin x}$

$\equiv$ RHS

5. Prove:

a) $\frac{\sin A}{1+\cos A}+\frac{1+\cos A}{\sin A}=\frac{2}{\sin A}$

b) Hence evaluate $\frac{\sin 2007°}{1+\cos 2007°}+\frac{1+\cos 2007°}{\sin 2007°}$

Solutions

a) $\frac{\sin A}{1+\cos A}+\frac{1+\cos A}{\sin A}=\frac{2}{\sin A}$

LHS $\frac{\sin^2 A+(1+\cos A)^2}{\sin A(1+\cos A)}$

$=\frac{\sin^2 A+1+2\cos A+\cos^2 A}{\sin A(1+\cos A)}$

$=\frac{2+2\cos A}{(\sin A)(1+\cos A)}$

$=\frac{2(1+\cos A)}{\sin A(1+\cos A)}$

$=\frac{2}{\sin A}=\text{RHS}$

b) We proved $\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x}=\frac{2}{\sin x}$

so $\frac{\sin 2007°}{1+\cos 2007°}+\frac{1+\cos 2007°}{\sin 2007°}$

$=\frac{2}{\sin 2007°}$

$=-4{,}405$

Activity

1. Simplify $\frac{\tan^2\theta-1}{\cos^2\theta}$
2. Simplify $1 \circ \sin^2\theta+\cos^2\theta$
3. Simplify $\frac{1}{\sin A}-\frac{\cos A}{\tan A}$
4. Simplify $\tan^2\alpha\cos^2\alpha+\frac{\sin^2\alpha}{\tan^2\alpha}$
5. Prove $\frac{\sin A-\sin^3 A}{\cos A-\cos^3 A}=\frac{1}{\tan A}$
6. Prove $\sqrt{1+2\sin\theta\cos\theta}=\sin\theta+\cos\theta$
7. Prove $\frac{1-\tan x}{1+\tan x}=\frac{\cos x-\sin x}{\cos x+\sin x}$
8. Prove $\frac{\tan^2\alpha}{1+\tan^2\alpha}=\sin^2\alpha$
9. Prove $\left(\frac{1}{\sin x}-\sin x\right)^2=\frac{1}{\tan^2 x}-\cos^2 x$
10. Prove $\frac{1-2\cos^2\theta}{\cos\theta\sin\theta}=\tan\theta-\frac{1}{\tan\theta}$
11. Prove $\frac{1}{\sin^2 A}-\frac{1}{\tan A\sin A}=\frac{1}{1+\cos A}$
12. Prove $\tan^2\alpha \circ \sin^2\alpha=\tan^2\alpha\times\sin^2\alpha$
13. Prove $\frac{\sin^2\alpha}{1+\cos\alpha}=1-\cos\alpha$
14. Prove $\frac{\sin x}{\cos x}=\frac{1+\cos x}{\sin x}$